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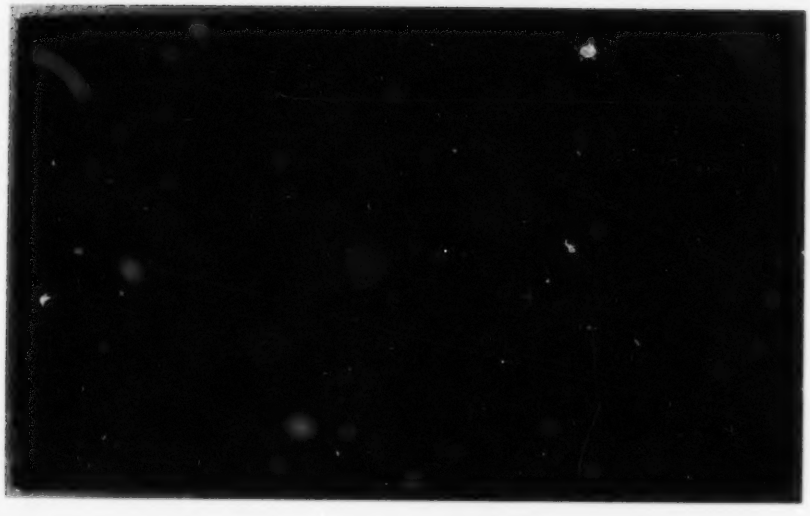

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System Capacity of a Road Subnetwork*

by

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ABSTRACT

This paper presents the concept of capacity of a road subsystem, it is an extension of capacity of a road and of an intersection in traffic engineering. Two mathematic models are presented. One is a vertex constrained max-flow problem, another is a Linear Programming problem. Solution method is discussed for both two models. The vertex constrained max-flow problem can be solved by an improved Ford-Fulkerson max-flow algorithm. The Linear Programming problem, because of its specific structure, could be solved by a two-phase algorithm to reduce its complexity of computation. The paper concludes with some possible research topics.

Key words : Road Network; Capacity.

RESUME

Ce texte présente le concept de capacité d'un sous-système routier. Le problème réside en une extension de capacité d'une route et d'une intersection en génie des transports. Deux modèles mathématiques sont présentés. Le premier est un problème de flot maximum contraint par un sommet. Le second est un problème de programmation linéaire. Une méthode pour résoudre ces problèmes est présentée pour les deux modèles. Le problème de flot maximum contraint par un sommet peut être résolu par une amélioration de l'algorithme du flot maximum de Ford-Fulkerson. Le problème de programmation linéaire, étant donné sa structure spécifique, peut être résolu par un algorithme en deux étapes, permettant ainsi de réduire la complexité des calculs. Le texte conclut sur un agenda de recherche.

Mots-clés: Réseau routier, capacité.

SYSTEM CAPACITY OF A ROAD SUBNETWORK

1. Introduction

Urban transportation, especially in large cities, is a complicated system affected by various socio-economic phenomena: population, land use activity, transportation infrastructure, vehicle ownership, transportation management, etc.. These factors restrict and impact on each other's development. If they develop at the same pace as the economy of the city, then transportation problem is avoidable. To guarantee socio-economic prosperity of the city, the urban transportation system must have sufficient transportation utilities, vehicles, roads and parking lots etc, to meet the needs of transportation demand. The transportation infrastructure and the number of vehicles, with transportation management, form the road transportation capacity of the city.

When for a city the population and functioning land use activity are fixed, the vehicle and passenger trips within the city are almost fixed by the development level of science and technology. But this not possible as population and land use are constantly changing, so we always have more or less problems on transportation.

Obviously the capacity of a road transportation system cannot be increased arbitrarily. First the transportation infrastructure cannot be developed indefinitely. It is subject to the constraints of urban space and economic support. Secondly under the wishing fixed transportation infrastructure it is not always possible to increase the transportation capacity by increasing the number of vehicles, because in any network system (gas network, water supply network, electricity distribution network and communication network, for example), there is an up bound, or criteria point, to its capacity, so does in transportation network system. The criteria point of the transportation network capacity is, generally speaking, the maximal number of vehicles travelling through the system within a given unit of time. When this point is surpassed, the system will run abnormally and transportation congestion will occur.

Unfortunately in most large cities of the world, there exist an imbalance between land use and transportation, that gives rise to various transportation problems[see Thompson 1972]. The problem is particularly severe in the large cities of China[Kajima,1987]. The main reason lies in the shortage of supply to and the excess of demand for transportation.

Let us take into account the road area in total land use, which is a basic indicator for evaluating urban transportation. Ideally road area should be 20 to 30 percent of total land use [Thompson,1972]. The road area for London, Paris and New York are 20, 20 and 30 percent respectively of the total land use, while the figures for Beijing, Tianjing and Shanghai are 3.3, 4.7 and 9.2 percent respectively[Kajima, 1987 and Thompson 1972]. The three largest cities of China have a road area rate that is half that of major international cities. Taking population into account, the road area per person in Shanghai is less than three square meters [SCTPI, 1990].

Unlike many developed countries, people travelling in the urban areas of China have only a choice between walking, cycling and taking a bus(at the end of 1989, there are about 200,000 motorized vehicles but 6,000,000 bicycles in Shanghai). There is no widespread train and subway system and highway system. There is a 46 kilometres subway system running in Beijing, and a 14.57 kilometres subway is under construction in Shanghai. The urban road system of China is therefore characterized by mixed traffic, thus increasing congestion problems.

This paper defines and discusses the capacity of a road system. The capacity of a road is the maximal number of vehicles passing through the cross section of the road in a unit of time, while the capacity of an intersection is the one relevant to the intersection [Paquette,1982]. Blunden and Black discussed the 'network capacity problem' to examine the effect of street closures on the though-traffic capacity of a network[Blunden and Black, 1984]. They treated the problem with one origin and one destination, and all traffic flow is from the origin to the destination. But in a real transportation system the flow on a road is often two ways rather than one way. We will, in this paper, extend the concept of 'network capacity problem' of Blunden and

Black, to consider the capacity problem with more than one origin and one destination. We will then provide the mathematical model for the problem and discuss the solution method.

The rest of the paper is organized as follows. Section 2 gives out some basic concepts and definitions. Section 3 generates mathematical models. Section 4 discusses the solution method. Section 5 concludes with some possible further research.

2. Basic Concepts and Definitions

2.1 Road subnetwork and its capacity

We consider a road subnetwork of an urban road system. A road subnetwork comprises road sections and intersections which form a connected network.

Assuming that there are k roads r_1, r_2, ... r_k, each r_i (i=1, 2, ... , k) intersects with several others, forming m intersections c_1, c_2, ... , c_m and n segments e_1, e_2, ... , e_n, these m intersections and n segments make up a road subnetwork S, that is connected.The capacity of the road subnetwork is defined as the maximal number of vehicles driving through S in unit of time, where a vehicle driving through S means that the vehicle enters into S from any inlet of S, passes through some intersections and segments in S and finally goes out of S (see Fig. 1). The capacity of S is expressed by F_s.

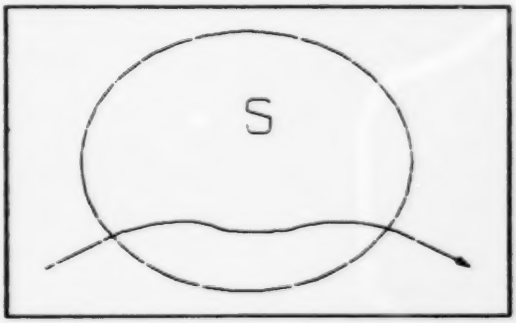


Figure 1. A vehicle drives through a road subnetwork S.

To make the above definition a sound one, we need an hypothesis. We suggest first, there is no traffic generator and/or absorber(parking lot, tunnel etc.) in S. This hypotheses seems to limit the practice of the definition, but we may see latter that the constraint can be relaxed.

2.2 Graph model of a road subnetwork

Road segments are generally divided into two sorts, i.e., an east-west direction and a south-north direction. For any segment two links, upward and downward, can be defined. For reason of simplicity we first take into account a situation where for any intersection in S, there is at most four segments connected to it. Under this case, the road subsystem can be seen as a grid network consisting of several horizontal roads and vertical roads (see Fig. 2).

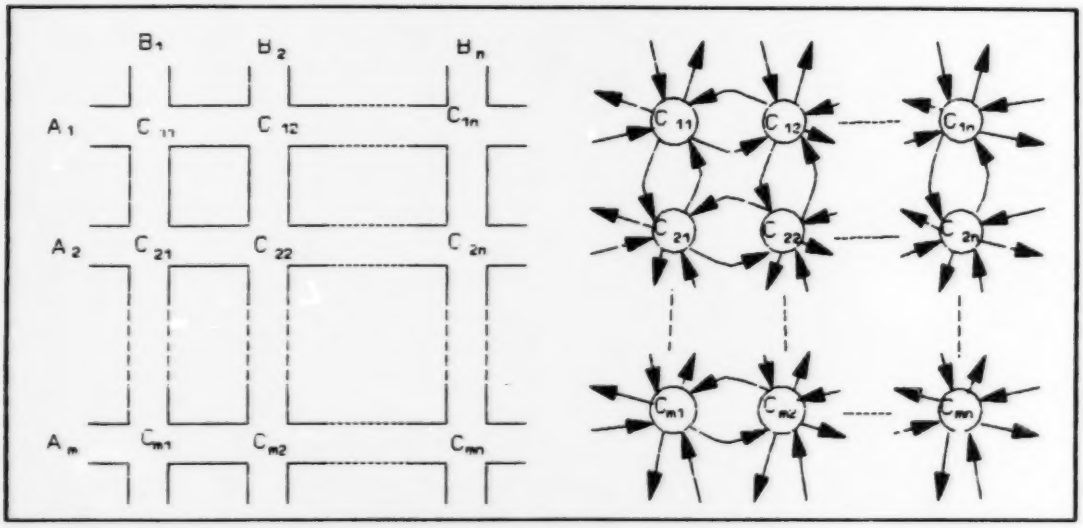


Fig. 2 A grid road network

Fig. 3 The graph model of the road subsystem shown in Fig. 2

Let $A_1, A_2, \ldots, A_m$ be m horizontal roads and $B_1, B_2, \ldots, B_n$ be n vertical roads, A_i intersects B_j at intersection C_{ij} (i=1, 2, ... , m; j=1, 2, ... , n), form a road subsystem S shown in Fig. 2. We use vertices to express the intersections, input origins and output destinations of S, use arcs to represent the links on roads, then the road system S can equally be drawn by a directed graph shown in Fig. 3.

By term of Graph Theory[see, for example, Bondy and Murty 1976; Swamy and Thulasiraman 1981], $V=\{v_1, v_2, \ldots, v_p\}$ is the set of all vertices and $A=\{a_1, a_2, \ldots, a_q\}$ is the set of all arcs. An arc a_i, which has an orientation directed by an arrow from v_i to v_j,

can be expressed by a pair of ordered vertices $a_i=(v_i, v_j)$, then v_i and v_j are called the initial and terminal vertex respectively of arc a_i. Note that arcs are oriented, (v_i, v_j) differs from (v_j, v_i).

An arc is said to be incident out of its initial vertex and incident into its terminal vertex, and the arc is called an out-arc of the initial vertex and in-arc of the terminal vertex respectively. An arc a' is said to be a descendant of the arc a=(i, j) if the terminal vertex of arc a' is the initial vertex of arc a, but not the (j, i); and if a' is a descendant of arc a then a is said to be an ancestor of a'.

In the graph-expressed road system, the vertices are divided into three types: input, output and interior. If there is no arc being incident into the vertex v_i, then v_i is said to be an input vertex, if there is no arc being incident out of v_i, then v_i is said to be an output vertex. A vertex being neither input nor output is an interior vertex. An arc is said to be an input arc if the arc is incident out of an input vertex, and an output arc if it is incident into an output vertex, and an internal arc if it is neither input nor output.

Let I represents the subset of V which comprises all input vertices of V, O represents the subset of V which comprises all output vertices of V, and N be the subset containing all interior vertices of V. Obviously V=I+O+N. And the set of all input arcs is expressed by A_I, the set of all output arcs A_O, and A_N the set of all internal arcs.

2.3 Probability of direction change

Let us suppose that a vehicle travels on link (v_i, v_j) in road subnetwork S, when it goes to the intersection v_j the vehicle can proceed by three different directions(in a grid network): turn left, go straight forward and turn right. Let l_{ij} express the probability of left-turn, in intersection v_j, of a vehicle travelling on link (v_i, v_j), s_{ij} be the probability of straight forwardness and r_{ij} the probability of right-turn. Because of no traffic absorber in S, the vehicle flow is conservative, i.e.,

$$l_{ij}+s_{ij}+r_{ij}=1, (i,j)\in A\backslash A_O$$

3. Mathematical Model

3.1 Mathematic model

Take c_{ij} as the capacity of the link $(i,j), (i,j) \in A$, and c_k as the capacity of the intersection v_k, $v_k \in N$. In transportation engineering the capacity of an intersection v_k, c_k, is a function of the link capacities of relevant roads which are incident to v_k, the duration of the traffic light and the rate of green lights in v_k. In the mathematical model of road subsystem capacity given below, we treat both link capacity and intersection capacity as exogenous constants.

Let f_{ij} represent the traffic volume on link (v_i, v_j). For input arc (v_i, v_j), f_{ij} is a decision variable in our mathematical model, while traffic volume on a non-input arc can be calculated by the ones on input arcs and probabilities of direction change. Let (v_i, v_j) be a non-input arc, and (v_l, v_i), (v_s, v_i) and (v_r, v_i) are descendant arcs of (v_i, v_j)[see Fig.4], then

$$f_{ij} = l_{li} f_{li} + s_{si} f_{si} + r_{ri} f_{ri}$$

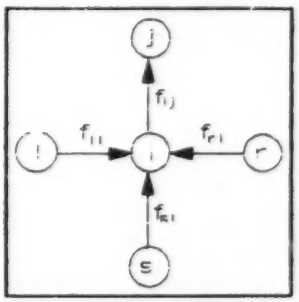


Figure 4 Illustration of calculation of traffic volume on a non-input arc

for a group of feasible variables $f_{ij}, 0 \le f_{ij} \le c_{ij}, (i,j) \in A$, $F = \sum_{(i,j) \in A_I} f_{ij}$ is just the number of vehicles passing through S in unit of time. In order to obtain the capacity of S, we should determine f_{ij} for every input arc (v_i, v_j) and then calculate f_{ij} for non-input arcs, subject to the constraints of both link capacities and intersection capacities, to make the summation F maximal.

The probabilities of direction change(PDC), l_{ij}, s_{ij} and r_{ij}, could be either decision variables or given constants. First we provide the mathematical model for road subnetwork with treating PDC as decision variables.

Let F_s represent the capacity of road system S, and F_s can be obtained from the solution of the following non-linear programming problem.

(SCPD)

$$F_s=\max \sum_{(i,j)\in A_I} f_{ij} \tag{3.1}$$

subject to

$$f_{ij}\le c_{ij},\ (i,j)\in A \tag{3.2}$$

$$\sum_{(i,k)\in A} f_{ik}\le c_k,\ k\in N \tag{3.3}$$

$$f_{ij}=l_{ii}f_{ii}+s_{si}f_{si}+r_{ri}f_{ri},\ (i,j)\in A\backslash A_I \tag{3.4}$$

$$l_{ij}+s_{ij}+r_{ij}=1,\ (i,j)\in A\backslash A_O \tag{3.5}$$

$$l_{ij},s_{ij},r_{ij}\ge 0,\ (i,j)\in A\backslash A_O \tag{3.6}$$

$$f_{ij}\ge 0,\ (i,j)\in A \tag{3.7}$$

where

f_{ij}: traffic volume on link (i, j)
c_{ij}: the capacity of link (i, j)
c_k : the capacity of intersection(vertex) v_k
l_{ij}: the probability of left-turn on link (i, j)
s_{ij}: the probability of straightness on link (i, j)
r_{ij}: the probability of right-turn on link (i, j)

constraints (3.2) and (3.3) express that the traffic flow cannot exceed the link capacity and intersection capacity; constraints (3.4) and (3.5) express that the traffic flow is conservative. Some assumptions are implied in (SCPD):

A1: No traffic generator and/or absorber exist in S;
A2: The decision variables f_{ij} vary continuously; and

A3: For a group of given c_{ij}, c_k, $(i,j)\in A$, $k\in N$, the PDC is decision variables, that means it is calculated by a system-optimal pattern.

System-optimal principal is plausible in (SCPD) when it is used to estimate the theoretical capacity of S and to give out an up-bound to the system under given constraints(c_{ij} and c_k are all constants). But for a practical real problem, say for a given O-D matrix pattern, (SCPD) can not directly be employed to analyze system capacity and guide transportation management and construction, because a specific O-D matrix pattern will produce a specific traffic flow on road system, the PDC is almost completely determined by traffic flow and the rule of transportation management. Under this condition a suitable mathematical model should treat the PDC as constant. The model can be stated as follows.

(SCPC)

$$\max \sum_{(i,j)\in A_I} f_{ij} \tag{3.1}$$

subject to

$$f_{ij}\le c_{ij},\ (i,j)\in A \tag{3.2}$$

$$\sum_{(i,j)\in A} f_{ik}\le c_k,\ k\in N \tag{3.3}$$

$$f_{ij}=l_{ii}f_{ii}+s_{si}f_{si}+r_{ri}f_{ri},\ (i,j)\in A\setminus A_I \tag{3.4}$$

$$f_{ij}\ge 0,\ (i,j)\in A \tag{3.7}$$

where

$$\begin{gathered} l_{ij}+s_{ij}+r_{ij}=1 \\ l_{ij},s_{ij},r_{ij}\ge 0,\ (i,j)\in A\setminus A_O \end{gathered} \tag{3.8}$$

Assumptions A1 and A2 in (SCPD) is still needed here, but A3 is substitute with a new one A3'.

A3': For a group of given c_{ij}, c_k, $(i,j) \in A$, $k \in N$, the PDC is given constants calculated by OD matrix.

3.2 Discussion on assumption and relaxation

Assumption A1 states that there exists no traffic generator and/or absorber in road system S, thus demanding that every vehicle travelling in S must have entered into it from its input links and should get out of the system by its output links, guaranteeing that the vehicle flow in S is conservative. Constraints (3.4) and (3.5), in fact, reflect the point in a grid network. If we remove simultaneously the condition that the road system is grid and the assumption A1, then the mathematical model (SCPD) and (SCPC), with a little change in (3.4) and (3.5) and (3.8) respectively, can also be applied, by following discussion, to express the general situation(arbitrary network with or without traffic generator and/or absorber).

Let (i, j) be a non-input arc, and (j_1, i), (j_2, i), ... , (j_s, i) are all of the descendant arcs of (i, j) with the PDC, $P_{(j_k, i)(i, j)}$, from (j_k, i) to (i, j), k=1, 2, ... , s; then we have[see Fig. 5(a)]

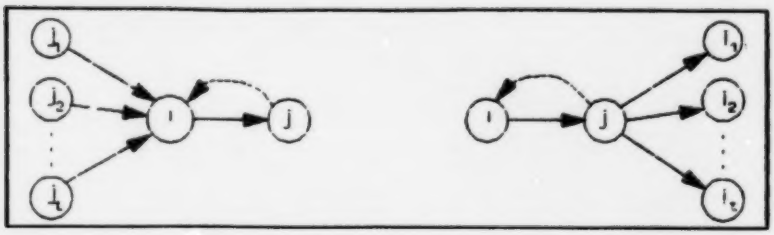
$$f_{ij} = \sum_{k=1}^{s} f_{j_k i} P_{(j_k, i)(i, j)} \qquad (3.4')$$

if (i, j) is a non-output arc and (j, i_1), (j, i_2), ... , (j, i_t) are all of the ancestors of (i, j) with the PDC, $P_{(i, j)(j, i_k)}$, from (i, j) to (j, i_k), k=1, 2, ... , t; then it holds [see Fig. 5(b)]

$$\sum_{k=1}^{t} P_{(i,j)(j,i_k)} = 1 \qquad (3.5')$$

$$P_{(i,j)(j,i_k)} \geq 0 \qquad (3.6')$$

Substituting (3.4'), (3.5') and (3.6') with (3.4), (3.5) and (3.6) in (SCPD), and substituting (3.5') and (3.6') with (3.8) in (SCPC), then (SCPD) and (SCPC) can work in an environment of arbitrary network and with traffic generators and/or absorbers.



(a) (b)

Figure 5 (a): arc (i, j) with more than three descendant arcs;
(b): arc (i, j) with more than three ancestors

Assumption A2 states that the variables f_{ij} vary continuously. It is unchangeable because f_{ij} is a sum of several production of $f_{j_k i}$ by $P_{(j_k,i)(i,j)}$ where the $P_{(j_k,i)(i,j)}$ are decimal.But in the next section we will see that under the condition that the PDC are decimal the solution of (SCPD), f_{ij}, can still be integers as long as all c_{ij}, c_k are integers.

Assumptions A3 and A3' states different optimal principles. Both of them need a prerequisite that the capacities of roads and intersections are known constants. Indeed, the capacity of intersection v_k, c_k, should be a function of the capacities of the relevant roads, the duration of traffic light and the rate of green lights on v_k. For example we can use the formulation

$$C_{AB}=\frac{a}{T}C_A+\frac{b}{T}C_B+e \qquad (3.9)$$
$$T=a+b+2y$$

where

C_A: the capacity of the road A;
C_B: the capacity of the road B;
C_{AB}: the capacity of the intersection where the road A directly meets the road B;
a: green time of road A;
b: green time of road B;
e: a constant;
y: yellow time for both road A and road B; and
T: traffic light period.

One can never take c_i as decision variables. Otherwise, following the definition of system capacity and equation (3.9), then the 'optimal' capacity of C_{AB} should satisfy the equation

$$C_{AB}=\max\left\{\frac{a}{a+b+2y}C_A+\frac{b}{a+b+2y}C_B+e \mid a+b+2y=T\right\} \qquad (3.10)$$

by choosing a and b. It immediately deduces, when $C_A \neq C_B$, that one equals zero and another is equal to T. This is trivial. Due to this reason we always take c_i as given constants.

4. Solution Method

4.1 Solution method for (SCPD)

4.1.1 vertex division

We have seen in section 3.1 that the constraints (3.4) are a non-linear equations, it seems difficult to solve (SCPD) with exact algorithm. While after changing the structure of graph model, the problem (3.1) through (3.7) becomes maximal flow style and can be easily solved.

Let us recall that the flow is supposed to be conservative in S. In fact, for every intersection v_i we have[see Fig.5(a) and equation (3.4')]

$$\sum_{(i,j)\in A} f_{ij}=\sum_{(i,j)\in A}\sum_{k=1}^{n} f_{j_k i}P_{(j_k,i)(i,j)} \qquad (4.1)$$

due to (3.5'), (4.1) becomes

$$\sum_{(i,j)\in A} f_{ij}=\sum_{(k,i)\in A} f_{ki} \qquad (4.2)$$

(4.2) is the flow conservative condition on an interior node in Maximal Flow Problem(MFP). The discrepancy between (SCPD) and (MFP) is, in (MFP) when the flow of (i, j) travelling to v_j, it can proceed by all arcs being incident out of v_j. In (SCPD), however, it can proceed all arcs except (j, i). So maximal flow algorithm can not directly be used to solve (SCPD). To make it usable we do the following vertex division.

For every interior vertex(take it as a crossroad for simplicity) v_i in S, v_i is divided into eight dots(see Fig.6): four in-dots(each is incident to an in-arc and three out-arcs) and four out-dots(each is incident to an out-arc and three in-arcs). The real effectiveness of this vertex division is to build a grade separation bridge, the flow on arc (i, j) cannot directly come back to arc (j, i).

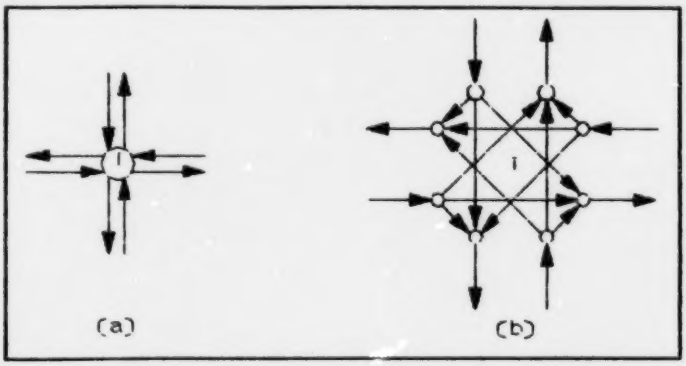


Figure 6 (a): a crossroad v_i
(b): vertex division of v_i

The dots divided from vertex v_i are grouped to D_i, we also call a dot an interior of S. An arc is said to be a dot-arc if its two terminal vertex are all in D_i, and A_i is the arc subset satisfying

$$A_i=\{(v_p, v_q) \mid v_p, v_q \in D_i\} \tag{4.3}$$

an arc is a regular arc if it is not a dot-arc. For a dot-arc (v_p, v_q) its capacity is determined by

$$c(v_p, v_q)=\min[c(v_{p_0}, v_p), c(v_q, v_{q_0})] \tag{4.4}$$

v_{p_0} and v_{q_0} are definitely determined in S by v_p and v_q respectively. Now the (SCPD) can be restated as follows.

(SCPD')

$$\max \sum_{(i,j)\in A} f_{ij} \tag{4.5}$$

subject to

$$\sum_{(i,j)\in A} f_{ij} = \sum_{(i,j)\in A} f_{jk},\ j\in N \tag{4.6}$$

$$0\le f_{ij}\le c_{ij},\ (i,j)\in A \tag{4.7}$$

$$\sum_{(p,q)\in A_j} f_{pq}\le c_j,\ j\in N \tag{4.8}$$

(SCPD') is much similar to the (MFP). In fact (4.5) through (4.7) is really the formulation of (MFP).

4.1.2 Algorithm(Ford and Fulkerson, 1962)

S1: Choose an initial feasible flow in S, generally set $f_{ij}=0$ for all $(i,j)\in A$.

S2: Search an augmentation path from an input vertex to an output vertex with regard to f. If do, turn to S3, if don't, algorithm terminates, it does so at optimal flow.

S3: Augment flow f along augmentation path, go to S2.

S2 should be renovated from the basis of Ford and Fulkerson algorithm because there exists the vertex capacity constraints in (SCPD).

S2: Search an augmentation path

Present flow f, vertex volume $f_k=\sum_{(i,j)\in D_k} f_{ij}$

S21: Labelling an unlabelled input vertex, if no, algorithm terminates, no augmentation path regarding to f exists.

S22: For an arc (i, j), if i is labelled and j is not, then label j if one of the following condition holds.

(a) $f_{ij}<c_{ij}$ for a regular arc of A;

(b) $f_{ij}< c_{ij}$ and $f_k<c_k$ for a dot-arc of A_k; and

(c) $f_{ij}<c_{ij}$ and $f_t=c_t$, but there exists a dot j_0 such that $f_{j\,j}>0$.

In the case of (c) j_0 should be labelled as well.

For an arc (i, j), if j is labelled and i is not, label i if $f_{ij}>0$.

S23: If an output vertex is labelled, an augmentation path is found, labelling process ends; if no more vertex can be labelled but all output vertex is unlabelled, initial the label for all vertices except the input ones and go to S21.

4.2 Methods for solving (SCPC)

(SCPC), as we have seen in section 3.1, is a Linear Programming model, one can solve it with a Simplex Method algorithm without any difficulty. What we can say here is that because of the specific structure of (SCPC), it can be more effectively solved by a two phases algorithm.

To simply analyze, but without loosing any generality, let the road system be a grid, with m horizontal roads and n vertical roads.This produces mn intersections and 2[m(m+1)+n(n+i)] links among which 2(m+n) links are input arcs. In (SCPC), there are 2[m(m+1)+n(n+1)] variables and, disregarding the up bound constraints and non-negative constraints on arcs, mn+2[m(m+1)+n(n+1)] constraint equations.But only those variables representing input arcs appear in the objective function. Therefore we can first, by the theorem stated below, calculate the expression of f_{ij} for all non-input arcs in term of the volumes of input arcs, thus largely decreasing the variables and constraints in (SCPC).

We start with some graph-theory terminology. A vertex adjacency matrix of a weighted directed graph G is

$$X(G)=[x_{ij}] \tag{4.9}$$

where

$$x_{ij}=\begin{cases} w_{ij}, & \text{if } (i,j)\in A(G) \text{ and } w_{ij} \text{ is the weight of } (i,j) \\ 0, & \text{otherwise.} \end{cases} \tag{4.10}$$

Similarly we define the arc adjacency matrix of a directed road system graph S as

$$\bar{X}=[\bar{x}_{ij}] \tag{4.11}$$

where

$$\bar{x}_{ij}=\begin{cases} P_{ij}, & \text{if arc } a_i \text{ is a descendant of arc } a_j \text{ and } P_{ij} \\ & \text{is the PDC from } a_i \text{ to } a_j; \\ 0, & \text{otherwise.} \end{cases} \tag{4.12}$$

if the arcs are arranged by order of input arcs, internal arcs and output arcs, then

$$\bar{X} = \begin{array}{c} \\ A_I \\ A_N \\ A_O \end{array} \begin{array}{c} \begin{array}{ccc} A_I & A_N & A_O \end{array} \\ \begin{bmatrix} 0 & C & 0 \\ 0 & B & D \\ 0 & 0 & 0 \end{bmatrix} \end{array} \tag{4.13}$$

the constraint (3.4') can be rewritten as

$$\bar{X}^t F=F$$

that is

$$\begin{bmatrix} 0 & 0 & 0 \\ C^t & B^t & 0 \\ 0 & D^t & 0 \end{bmatrix} \begin{bmatrix} F_I \\ F_N \\ F_O \end{bmatrix} = \begin{bmatrix} F_I \\ F_N \\ F_O \end{bmatrix} \tag{4.14}$$

here F is a vector of flow on arcs, F has the same arcs' order as X does, and F_I corresponds to input arcs, F_O corresponds to internal arcs and F_N corresponds to internal arcs. From (4.14) we have

$$\begin{aligned} D^t F_N &= F_O \\ (I-B) F_N &= C^t F_I \end{aligned} \tag{4.15}$$

A road system graph G is said to be strongly connected with regard to PDC, if for every pair of ordered vertices v_i and v_j of G, there exists a directed path from v_i to v_j such that the product of the probabilities of PDC along this path is not zero. The subgraph of G generated by A_N is called the base graph of S.

THEOREM 1 Suppose X is the weighted arc adjacency matrix of a directed road system graph, X is written by form (4.13), then the matrix (I-B) is non-singular if and only if $D \neq 0$.

Proof: Recall that for PDC it holds

$$\sum_{(i,j)\in A} P_{(i,j)(j,k)}=1 \qquad (3.5')$$

say

$$\sum_{j=1}^{\gamma} B_j+\sum_{j=1}^{\beta} D_j=1$$

where B_j is the jth column of B, D_j is the jth column of D and

$$\gamma=|A_N|,\ \beta=|A_O|$$

when D=0, it follows

$$\sum_{j=1}^{\gamma} B_j=1$$

immediately deducing

$$\det(I-B)=0$$

For sufficiency, we first suppose that for every row and every column of (I-B), there is at least one more non-zero entry other than 1 in diagonal, otherwise one can deal with a smaller graph derived by deleting from the original graph the arc corresponding to the row (or column) which has only non-zero entry 1 in diagonal. Under this condition the base graph of S is strongly connected with regard to PDC.

Now if there exists a group of numbers $y_1, y_2, \cdots, y_\gamma$ such that

$$\sum_{i=1}^{\gamma} y_i B_i'=0$$

B_i' is the ith column of matrix (I-B),we prove that $y_i=0$ for all i.

Because $D \neq 0$ there is at least one row in (I-B) such that the sum of entries in this row is not zero, let this row be the i_1th row. Let $y_{i_1} \geq 0$ (if $y_{i_1} < 0$, substitute y_i with $-y_i$ for all i), in order to make (4.17) holds, there must be an $i_2 \neq i_1$ such that $y_{i_2} \geq y_{i_1}$, and there exists $i_3 \neq i_2$ satisfying $y_{i_3} \geq y_{i_2}$ when we take the i_2th row into account, ... (see Fig. 7). Because the base graph is strongly connected with regard to PDC, it finally deduces

$$y_1 = y_2 = \ldots = y_v = 0$$

hence the theorem.

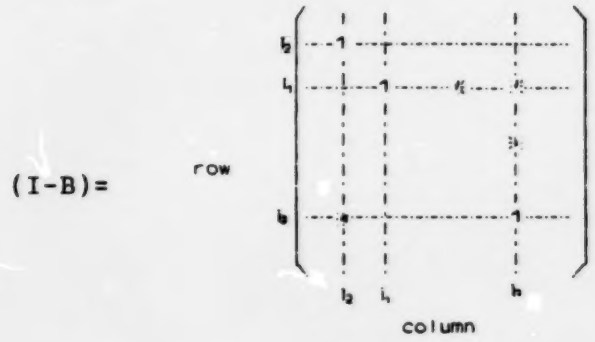


Figure 7 Illustration for theorem proof, * is a non-zero entry.

The determinant of matrix (I-B) can also be calculated by a graph-theory method other than algebraic approach.

Let v be a vertex of a directed graph G, the in-degree of v in G is the number of arcs being incident into v, and the out-degree of v is the number of arcs being incident out of v.

A 1-factor of a directed graph G is a spanning subgraph of G in which the in-degree and out-degree of every vertex are both equal to one. Following theorem is due to Harary[Harary, 1962].

THEOREM 2 The determinant of the adjacency matrix X of an n-vertex weighted directed connected graph G is given by

$$\det X = (-1)^n \sum_H (-1)^{L_H} \omega(H)$$

where H is a 1-factor of G, w(H) is the weight product of H and L_H is the number of directed circuits in H.

To apply theorem 2 to a base graph of a road system, first we generate an edge graph E^G, which takes the matrix B of G as the vertex adjacency matrix, theorem 2 can be used to compute the determinant of $X(E^G)=B$ as long as E^G is connected. In the case E^G is disconnected, theorem 2 should be employed to every connected components and the determinant of $X(E^G)$ is the product of all the determinants of the components.

Note that a circuit in E^G is one to one mapping to a closed arc-disjoint arcs sequence in G along which the PDC are not zero, call such a sequence a proper closed trial. So a 1-factor in E^G maps to a collection of arc-disjoint proper closed trials covering all arcs in G, and vice versa.

EXAMPLE A two by two road system is shown in Fig. 8(a), the number on the arcs is the code of that link, the capacity of arc a_i is c_i and the capacity of intersection k is c'_k.

the base graph of the road system is shown in Fig. 8(b), and Fig. 8(c) is the edge graph of (b). The determinant of G of (b) is given by

$$\det X(G) = (1-l_9 l_{10} l_{11} l_{12})(1-r_{13} r_{14} r_{15} r_{16})$$

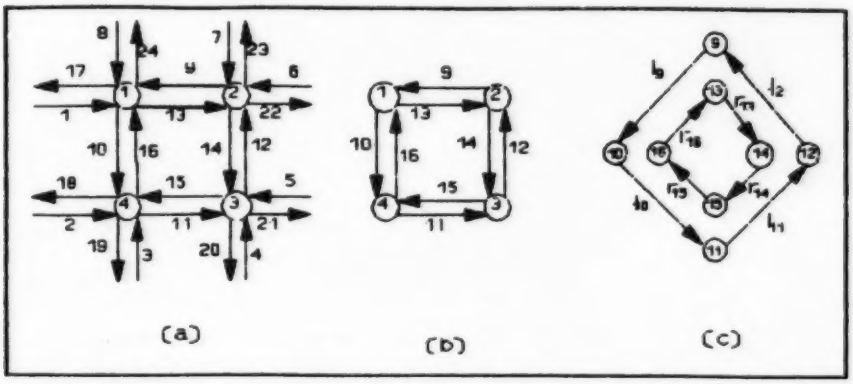


Figure 8: Demonstration example. (a) a 2 by 2 road system; (b) the base graph of (a);and (c) edge graph EG of G.

5. Conclusion and Further Research

This paper presents the concept of capacity of a road subsystem, it is an extension of capacity of a road and of an intersection in traffic engineering. Two mathematic models are presented. One is a vertex-constrained max-flow problem which treats the PDC as variables(SCPD), it can be solved by Ford-Fulkerson max-flow algorithm with a little change in vertex label process. Another is a Linear Programming problem in which the PDC is given constants(SCPC), because of its specific structure the problem could be solved by a two-phase algorithm, first to solve a group of linear equations to get an explicit expression of intermediate variables by input arcs and secondly to solve a linear programming problem.

(SCPD) can be employed to estimate the up bound of the capacity of a road system and (SCPC) can be used to analyze a practical problem-- capacity of real road system--with the PDC calibrated by field survey data.

Further research could be extended to three aspects. Firstly, we should gather relevant data and apply the theory to real problem in Shanghai. Secondly, PDC is definitely relevant with OD matrix pattern, so how to use OD pairs and traffic assignment technique to estimate PDC should be an interesting study. Thirdly, the concept of system capacity could possibly be introduced to traffic

management and vehicle navigation[Boyce, 1989]. The research to be undertaken includes prediction of congestion status in a road subsystem when the number of vehicles driving into the system in unit of time approaches or exceeds its capacity, estimation of relation among system capacity , total vehicle volume in the system and travel time delay and relationship between system capacity and traffic management.

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